2 The Mathematics of Power

Shapley-Shubik Measure of Power

A different approach to measuring power, first proposed by American mathematician Lloyd Shapley and economist Martin Shubik in 1954. The key difference between the Shapley-Shubik measure of power and the Banzhaf measure of power centers on the concept of a sequential coalition.

In the Shapley-Shubik method the assumption is that coalitions are formed sequentially:

Shapley-Shubik Measure of Power

Players join the coalition and cast their votes in an orderly sequence (there is a first player, then comes a second player, then a third, etc.).

Thus, to an already complicated situation we are adding one more wrinkle–the question of the order in which the players join the coalition.

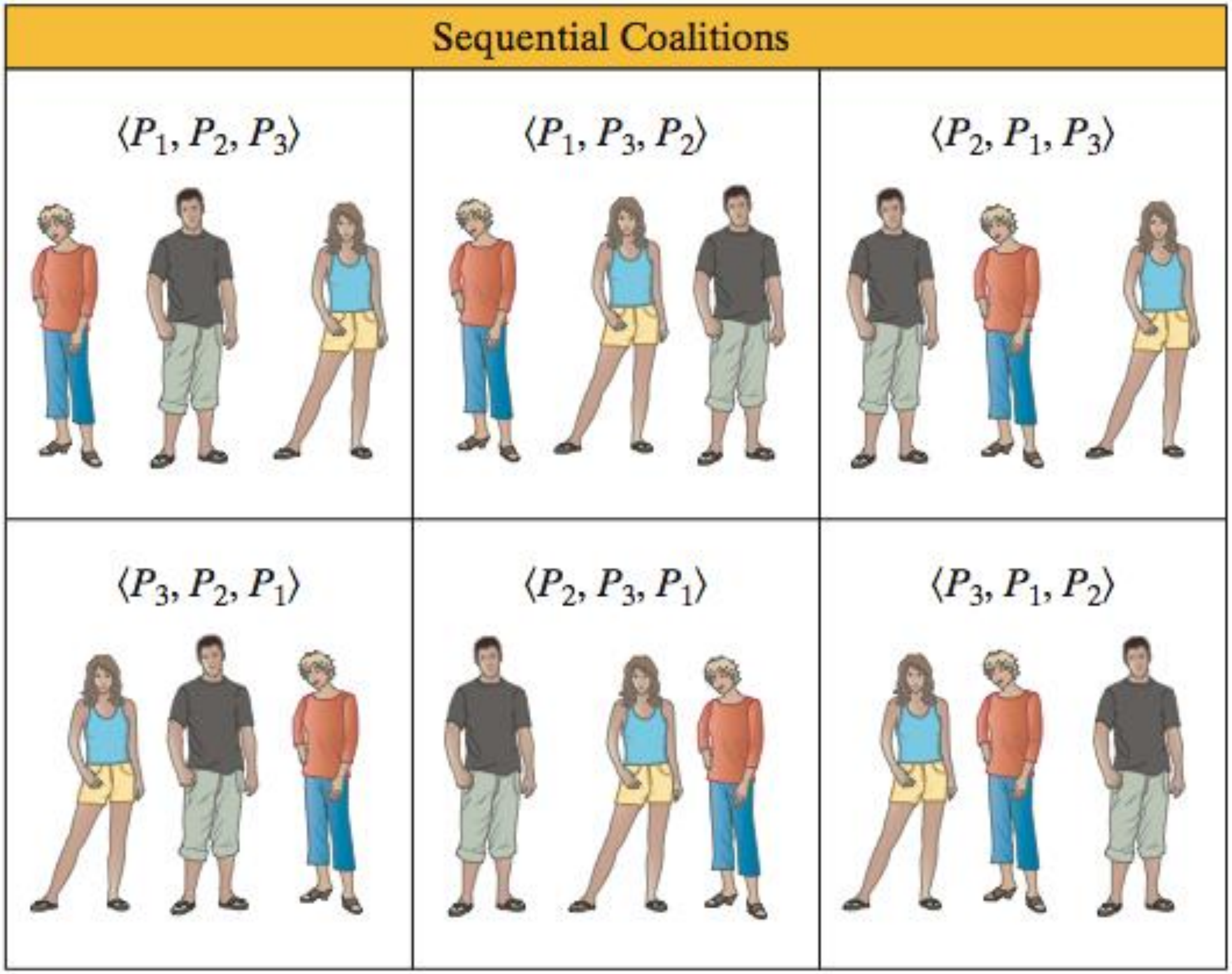
Ex 2.13 Three-Player Sequential Coalitions

When we think of the coalition involving three players P_1, P_2, and P_3, we think in terms of a set. For convenience we write the set as $\{P_1, P_2, P_3\}$, but we can also write in other ways such as $\{P_3, P_2, P_1\}$, $\{P_2, P_3, P_1\}$, and so on. In a coalition the only thing that matters is who are the members–the order in which we list them is irrelevant.

Ex 2.13 Three-Player Sequential Coalitions

In a sequential coalition, the order of the players does matter –there is a "first" player, a "second" player, and so on. For three players P_1, P_2, and P_3, we can form six different sequential coalitions: $\langle P_1, P_2, P_3 \rangle$, P is the first player, the P joined in, and last came P_3; $\langle P_1, P_3, P_2 \rangle; \langle P_2, P_1, P_3 \rangle; \langle P_2, P_3, P_1 \rangle;$ $\langle P_3, P_1, P_2 \rangle; \langle P_3, P_2, P_1 \rangle;$ as shown on the next slide.

Ex 2.13 Three-Player Sequential Coalitions



Shapley-Shubik Approach

Pivotal player. In every sequential coalition there is a player who contributes the votes that turn what was a losing coalition into a winning coalition–we call such a player the pivotal player of the sequential coalition. Every sequential coalition has one and only one pivotal player. To find the pivotal player we just add the players' weights from left to right, one at a time, until the tally is bigger or equal to the quota q. The player whose votes tip the scales is the pivotal player.

Shapley-Shubik Approach

The Shapley-Shubik power index. For a given player *P*, the Shapley-Shubik power index of *P* is obtained by counting the number of times *P* is a *pivotal* player and dividing this number by the total number of times *all* players are pivotal.

Shapley-Shubik Approach

The Shapley-Shubik power index of a player can be thought of as a measure of the size of that player's "slice" of the "power pie" and can be expressed either as a fraction between 0 and 1 or as a percent between 0 and 100%.

Shapley-Shubik Approach

The Shapley-Shubik power index.

We will use σ_1 ("sigma-one") to denote the Shapley-Shubik power index of P_1, σ_2 to denote the Shapley- Shubik power index of P_2, and so on.

Shapley-Shubik Approach

The Shapley-Shubik power distribution. The complete listing of the Shapley- Shubik power indexes of all the players is called the Shapley-Shubik power distribution of the weighted voting system.

It tells the complete story of how the (Shapley-Shubik) power pie is divided among the players.

Summary of Steps

COMPUTING A SHAPLEY-SHUBIK POWER DISTRIBUTION

Step 1. Make a list of all possible sequential coalitions of the *N* players. Let *T* be the number of such coalitions.

(We will have a lot more to say about *T* soon!)

Step 2. In each sequential coalition determine the pivotal player.

(For bookkeeping purposes underline the pivotal players.)

Summary of Steps

Step 3. Count the total number of times that P_1 is pivotal. This gives SS_1, the pivotal count for P_1. Repeat for each of the other players to find $SS_2, SS_3, \ldots, SS_N$.

Step 4. Find the ratio $\sigma_1 = SS_1/T$. This gives the *Shapley-Shubik power index* of P_1. Repeat for each of the other players to find $\sigma_2, \sigma_3, \ldots, \sigma_N$. The complete list of σ's gives the *Shapley-Shubik power distribution* of the weighted voting system.

Multiplication Rule

The Multiplication Rule is one of the most useful rules of basic mathematics.

THE MULTIPLICATION RULE

If there are m different ways to do X and n different ways to do Y, then X and Y together can be done in $m \times n$ different ways.

Ex 2.15 Cones and Flavors Plus Toppings

A bigger ice cream shop offers 5 different choices of cones, 31 different flavors of ice cream, and 8 different choices of topping. The question is, If you are going to choose a cone and a single scoop of ice cream but then add a topping for good measure, how many orders are possible?

(think back to algebra 2…Mr. potatohead)

Ex 2.15 Cones and Flavors Plus Toppings

The number of choices is too large to list individually, but we can find it by using the multiplication rule twice: First, there are $5 \times 31 = 155$ different cone/flavor combinations, and each of these can be combined with one of the 8 toppings into a grand total of $155 \times 8 = 1240$ different cone/flavor/topping combinations. The implications – as well as the calories – are staggering!

Ex 2.16 Counting Sequential Coalitions

How do we count the number of sequential coalitions with four players? Using the multiplication rule, we can argue as follows: We can choose any one of the four players to go first, then choose any one of the remaining three players to go second, then choose any one of the remaining two players to go third, and finally the one player left goes last. Using the multiplication rule we get a total of $4 \times 3 \times 2 \times 1 = 24$ sequential coalitions with four players.

Factorial

If we have five players, following up on our previous argument, we can count on a total of $5 \times 4 \times 3 \times 2 \times 1 = 120$ sequential coalitions, and with N players the number of sequential coalitions is $N \times (N - 1) \times \ldots \times 3 \times 2 \times 1$. The number $N \times (N - 1) \times \ldots \times 3 \times 2 \times 1$ is called the **factorial** of N and is written in the shorthand form $N!$.

Sequential Coalitions

THE NUMBER OF SEQUENTIAL COALITIONS

$N! = N \times (N - 1) \times \ldots \times 3 \times 2 \times 1$ gives the number of sequential coalitions with N players.

Ex 2.18 Shapley-Shubik Power and the NBA Draft

We will now revisit Example 2.10, the NBA draft example. The weighted voting system in this example is [6: 4, 3, 2, 1], and we will now find its Shapley-Shubik power distribution.

Step 1 and 2. Table 2-10 shows the 24 sequential coalitions of P_1, P_2, P_3, and P_4. In each sequential coalition the pivotal player is underlined. (Each column corresponds to the sequential coalitions with a given first player.)

Example 2.18 Shapley-Shubik Power and the NBA Draft

TABLE 2-10 Sequential Coalitions for [6: 4, 3, 2, 1] (Pivotal Players Underlined)

$\langle P_1, \underline{P_2}, P_3, P_4 \rangle$	$\langle P_2, \underline{P_1}, P_3, P_4 \rangle$	$\langle P_3, \underline{P_1}, P_2, P_4 \rangle$	$\langle P_4, P_1, \underline{P_2}, P_3 \rangle$
$\langle P_1, \underline{P_2}, P_4, P_3 \rangle$	$\langle P_2, \underline{P_1}, P_4, P_3 \rangle$	$\langle P_3, \underline{P_1}, P_4, P_2 \rangle$	$\langle P_4, P_1, \underline{P_3}, P_2 \rangle$
$\langle P_1, \underline{P_3}, P_2, P_4 \rangle$	$\langle P_2, P_3, \underline{P_1}, P_4 \rangle$	$\langle P_3, P_2, \underline{P_1}, P_4 \rangle$	$\langle P_4, P_2, \underline{P_1}, P_3 \rangle$
$\langle P_1, \underline{P_3}, P_4, P_2 \rangle$	$\langle P_2, P_3, \underline{P_4}, P_1 \rangle$	$\langle P_3, P_2, \underline{P_4}, P_1 \rangle$	$\langle P_4, P_2, \underline{P_3}, P_1 \rangle$
$\langle P_1, P_4, \underline{P_2}, P_3 \rangle$	$\langle P_2, P_4, \underline{P_1}, P_3 \rangle$	$\langle P_3, P_4, \underline{P_1}, P_2 \rangle$	$\langle P_4, P_3, \underline{P_1}, P_2 \rangle$
$\langle P_1, P_4, \underline{P_3}, P_2 \rangle$	$\langle P_2, P_4, \underline{P_3}, P_1 \rangle$	$\langle P_3, P_4, \underline{P_2}, P_1 \rangle$	$\langle P_4, P_3, \underline{P_2}, P_1 \rangle$

Step 3. The pivotal counts are $SS_1 = 10$, $SS_2 = 6$, $SS_3 = 6$ and $SS_4 = 2$.

Example 2.18 Shapley-Shubik Power and the NBA Draft

Step 4. The Shapley-Shubik power distribution is given by

$$\sigma_1 = \frac{10}{24} = 41\frac{2}{3}\% \qquad \sigma_2 = \frac{6}{24} = 25\%$$

$$\sigma_3 = \frac{6}{24} = 25\% \qquad \sigma_4 = \frac{2}{24} = 8\frac{1}{3}\%$$

Example 2.18 Shapley-Shubik Power and the NBA Draft

If you compare this result with the Banzhaf power distribution obtained in Example 2.10, you will notice that here the two power distributions are the same. If nothing else, this shows that it is not impossible for the Banzhaf and Shapley- Shubik power distributions to agree. In general, however, for randomly chosen real-life situations it is very unlikely that the Banzhaf and Shapley-Shubik methods will give the same answer.